$:=$ defined as

$\forall$: for all
$\exists$: there exists one, $\exists 1 = \exists +$ unique
$\ni$: such that.

Notations

- $a \in S$
- 函數 $f: \begin{array}{ccc} A & \longrightarrow & B \\ \Downarrow & & \Downarrow \\ x & \longmapsto & y \end{array}$ $\forall x \in A,\ \exists 1\ y \in B,\ \ni$ $f(x) = y$ (Image) $f(A) = \{ f(x) \mid x \in A \} \subset B$

  - 1-1 (injective) : $x \neq y \Rightarrow f(x) \neq f(y)$
  - onto (surjective) : $\forall y \in B,\ \exists x \in A,\ f(x) = y$ $(f(A) = B)$
  - 1-1, onto (bijective) :

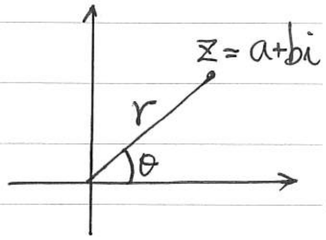
    $\Downarrow$

    $\exists f^{-1}: B \to A$ (inverse of f)

    $f(x) \mapsto x$

  - $\left\{ \begin{matrix} f \\ g \end{matrix} \right. : A \to B$ $\left\{ \begin{matrix} (f+g) \\ (cf) \end{matrix} \right. : A \to B,$ $\begin{matrix} x \longmapsto f(x)+g(x) \\ x \longmapsto c\, f(x) \end{matrix}$ $\left[\begin{matrix} (f+g)(x) = f(x)+g(x) \\ (cf)(x) = c f(x) \end{matrix} \right]$
  - $$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, dashed, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$ $gf(x) = g(f(x))$

- Integers : $\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$, $\mathbb{N} = \{ 1, 2, 3, \dots \}$ Natural no.
- Rational no. : $\mathbb{Q} = \{ \frac{q}{p} \mid p.q \in \mathbb{Z},\ p \neq 0 \}$
- Real " : $\mathbb{R}$
- Complex " : $\mathbb{C} = \{ a+bi \mid a, b \in \mathbb{R} \}$



$z = a+bi \in \mathbb{C}$, $\bar{z} = a-bi$ (conjugate) $\Rightarrow z\bar{z} = a^2+b^2 = r^2$

$|z| = r = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

$z = r(\cos\theta + i\sin\theta)$ (極坐標)

$= re^{i\theta}$

性質 (1) $\overline{(\bar{z})} = z$

(2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(3) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

Linear Space (平直空間)

Vector Space over $\mathbb{C}$: $V_{\mathbb{C}}$ (複係數向量空間)

滿足 $\forall \begin{cases} u, v \in V & \text{(向量)} \\ c, d \in \mathbb{C} & \text{(純量)} \end{cases}$

(0) $u + v \in V$ $\qquad$ $cu \in V$

(1) $u + v = v + u$ $\qquad$ (5) $c(u+v) = cu + cv$

(2) $(u+v) + w = u + (v+w)$ $\qquad$ (6) $(c+d)u = cu + du$

(3) $\exists\, 0 \in V,\ 0 + v = v$ $\qquad$ (7) $a(bu) = (ab)u$

(4) $\forall u,\ \exists(-u),\ u + (-u) = 0$ $\qquad$ (8) $1u = u$

推論 : $0u = 0$

$(\because\ cu = (c+0)u = cu + 0u)$

Example

(1) $\mathbb{C}_n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in \mathbb{C} \right\}$ $\qquad$ $\begin{cases} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} \\ c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix} \end{cases}$

(2) $\mathcal{H}_n = \left\{ a_0|0\rangle + a_1|1\rangle + \cdots + a_{n-1}|n-1\rangle \middle| a_i \in \mathbb{C} \right\}$ $\qquad$ $|0\rangle \neq 0$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(3) $P_n = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \middle| a_i \in \mathbb{C} \right\}$

- $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{n-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ $\quad \leftrightarrow a_0|0\rangle + a_1|1\rangle + \cdots + a_{n-1}|n-1\rangle$

$= a_0 \mathbb{e}_1 + a_1 \mathbb{e}_2 + \cdots + a_{n-1} \mathbb{e}_n$ $\quad \leftrightarrow a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$

$\mathbb{C}_n \cong \mathcal{H}_n \cong P_n$ isomorphic (同構) under basis

$\{\mathbb{e}_i\}_{1 \le i \le n}$ $\{|i\rangle\}$ $\{x^i\}$

Matrix

$$\begin{cases} \mathcal{M}_{m\times n} = \{A_{m\times n}\} \\ \mathcal{M}_{n} = \mathcal{M}_{n\times n} \end{cases}$$

$$\begin{array}{cc} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{array} & \\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & a_{ij} & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} & \begin{array}{l} \leftarrow A_1 \\ \leftarrow A_2 \\ \vdots \\ \leftarrow A_m \end{array} \end{array}$$

- Zero matrix $O_{m\times n}$
- Square 〃 $(m=n)$
  - diagonal 〃
  - upper triangular 〃
    lower 〃 〃
  - Identity 〃 $I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - Symmetric : $A^T = A$
  - Hermitian : $A^* = A$
  - Unitary : $A^*A = I_n$

Operations

$A = [a_{ij}]$, $B = [b_{ij}]$

- $A_{m\times n} = B_{m\times n}$ $a_{ij} = b_{ij}, \forall i,j$
- $cA_{m\times n} := [ca_{ij}]_{m\times n}$
- $A_{m\times n} \pm B_{m\times n} := [a_{ij} \pm b_{ij}]_{m\times n}$
- $A_{m\times p} B_{p\times n} := [c_{ij}]_{m\times n}$ $c_{ij} = c_{i1} b_{1j} + \cdots + c_{ip} b_{pj} = A_i \cdot b_j$
- $A^T_{m\times n} := [c_{ij}]_{n\times m}$ $c_{ij} = a_{ji}$ (transpose of A)
- $A^* = A^\dagger := (\bar{A})^T$ (Hermitian-adjoint of A)
- $A^0 = I$, $A^m = \underbrace{AA\cdots A}_{m\text{次}}$

Property

(1) $A + B = B + A$
$(A+B)+C = A+(B+C)$
$A + 0 = A$
$A + (-A) = 0$

(2) $c(dA) = (cd)A$
$c(A+B) = cA + cB$
$(c+d)A = cA + dA$
$1 \cdot A = A$

$(\mathcal{M}_{m\times n}$: Vector space

(3) $AI_n = A = I_m A$
$(A+A')B = AB + A'B$
$A(B+B') = AB + AB'$
$(cA)B = c(AB) = A(cB)$
$(AB)C = A(BC)$

(4) $AB \neq BA$

(5) $(A^T)^T$ $(A^*)^* = A$
$(A+B)^T = A^T + B^T$ $(A+B)^* = A^* + B^*$
$(cA)^T = cA^T$ $(cA)^* = \bar{c}A^*$
$(AB)^T = B^T A^T$ $(AB)^* = B^* A^*$

Matrix multiplication

$$\begin{array}{c} \begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ -1 & -2 & -3 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \\ -x-2y-3z \end{bmatrix} = x\begin{bmatrix} 1 \\ 4 \\ 7 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ 8 \\ -2 \end{bmatrix} + z\begin{bmatrix} 3 \\ 6 \\ 9 \\ -3 \end{bmatrix}$$

$$A \qquad \mathbf{x} \qquad\qquad = x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ -1 & -2 & -3 \end{bmatrix} \begin{array}{c} \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{matrix} \\ \begin{bmatrix} x & a & m \\ y & b & n \\ z & c & p \end{bmatrix} \end{array} = \begin{bmatrix} 1x+2y+3z & 1a+2b+3c & 1m+2n+3p \\ 4x+5y+6z & 4a+5b+6c & 4m+5n+6p \\ 7x+8y+9z & 7a+8b+9c & 7m+8n+9p \\ -x-2y-3z & -a-2b-3c & -m-2n-3p \end{bmatrix}$$

$$A \qquad B \qquad = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$$

Theorem (nonsingular)

$A_{n\times n}$

下列敘述等價，都可作為 nonsingular matrix 的定義

(1) $\operatorname{rank}(A) = \#\text{pivots} = n$

(2) $A\mathbf{x} = \mathbf{b}$ 有唯一解

(3) $A\mathbf{x} = \mathbf{0}$ 只有 0 解

(4) A^{-1} 存在

(5) $\det(A) \neq 0$

(6) $\{A_1, \cdots, A_n\}$ basis for $\mathbb{R}^n$

(7) $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ 〃

(8) $N(A) = \operatorname{Ker}(T_A) = \{\mathbf{0}\} \iff T_A$: 1-1

(9) $\operatorname{Im}(A) = \operatorname{Im}(T_A) = \mathbb{R}^n \iff T_A$ onto

(10) 0 不是 eigen value

- A invertible (可逆) $\overset{def}{\Longleftrightarrow}$ $\exists\, E,\ EA = AE = I_n$ $\quad (E = A^{-1})$

inverse (逆矩陣) of A

性質

A : invertible 則 (1) $\Leftrightarrow$ A nonsingular

B

(2) $A\mathbb{x} = \mathbb{b}$ 有唯-解 $\mathbb{x} = A^{-1}\mathbb{b}$

(3) E invertible 且 $E^{-1} = A$ 即 $(A^{-1})^{-1} = A$

(4) $(AB)^{-1} = B^{-1}A^{-1}$

(5) $(A^T)^{-1} = (A^{-1})^T$ Pf: $AA^{-1} = I \Rightarrow (A^{-1})A^T = I$

(6) $EA = I \Rightarrow AE = I$ 即 $\begin{cases} A^{-1} = E \\ E^{-1} = A \end{cases}$

Pf: 解 $\underset{E}{A}\mathbb{X} = \underset{E}{I} \Rightarrow \mathbb{X} = E$

- **Block partition of Matrix**

$$\underset{\substack{4\times 6 \\ [2\times 3]}}{\left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 2 & -1 & 1 \\ 3 & 1 & -1 & 3 & 2 & 0 \\ \hline 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 1 \end{array}\right]}
\underset{\substack{6\times 3 \\ [3\times 2]}}{\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & -3 & 1 \\ \hline -1 & 1 & -2 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \\ \hline 2 & 1 & 1 \end{array}\right]}
= \underset{\substack{4\times 3 \\ [2\times 2]}}{\left[\begin{array}{cc|c} 1+4+0+6-1+2 & 0-6+0+4+0+1 & 2 \\ 3+2+1\div 9+2+0 & 0-3-1+6+0+0 & 20 \\ \hline -2 & -1 & 5 \\ 5 & 0 & 10 \end{array}\right]}$$

$$= \left[\begin{array}{c|c}
\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} \\ \hline
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}
\end{array}\right]$$

$$= \left[\begin{array}{c|c}
\begin{bmatrix} 1+4 & 0-6 \\ 3+2 & 0-3 \end{bmatrix} + \begin{bmatrix} 0+6-1 & 0+4+0 \\ 1+9+2 & -1+6+0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 5 \\ 10 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \hline
\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{array}\right]$$

$$= \underset{\substack{4\times 3 \\ [2\times 2]}}{\left[\begin{array}{c|c}
\begin{bmatrix} 1+4+0+6-1+2 & 0-6+0+4+0+1 \\ 3+2+1+9+2+0 & 0-3-1+6+0+0 \end{bmatrix} & \begin{bmatrix} 2 \\ 20 \end{bmatrix} \\ \hline
\begin{bmatrix} -2 & -1 \\ 5 & 0 \end{bmatrix} & \begin{bmatrix} 5 \\ 10 \end{bmatrix}
\end{array}\right]}$$

Basis & Dimension

V : vector space over $\mathbb{C}$

- $\mathbb{U}$: subspace of V $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \text{(i)} & \mathbb{U} \subset V \\ \text{(ii)} & \mathbb{U} \text{ is vector space under } \begin{cases} + \\ \cdot \end{cases} \end{cases}$

  $(\mathbb{U} \triangleleft V)$

  $\Longleftrightarrow \begin{cases} (1) & \mathbb{0} \\ (2) & \mathbf{u}+\mathbf{v} \\ (3) & c\mathbf{v} \end{cases} \in \mathbb{U}\ ,\ \forall \begin{matrix} \mathbf{u}, \mathbf{v} \in \mathbb{U} \\ c \in \mathbb{C} \end{matrix}$

Example

(1) $\{\mathbb{0}\} \triangleleft V$

(2) $V \triangleleft V$

(3) $\mathrm{Span}(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \mid a_i \in \mathbb{C}\} \triangleleft V$,

註 $\begin{cases} \mathrm{Span}(\mathbf{v}_1) \text{ 直線} \\ \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \text{ 平面} \end{cases}$

(4) $\{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \mathbb{0}\} \triangleleft \mathbb{C}^n$

(5) $W \triangleleft \mathbb{C}^n \Rightarrow W^{\perp} = \{\mathbf{x} \in \mathbb{C}^n \mid \mathbf{x} \perp W\} \triangleleft \mathbb{C}^n$

- $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \subset V$ is linear dependent : $\exists \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbb{0}$

  (相依) (a_i 不全為 0)

  〃 independent : $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbb{0} \Rightarrow a_1 = a_2 = \cdots = a_k = 0$

  (獨立)

- $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ basis for $\mathbb{U}$ $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} (1) & \mathrm{Span}(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \mathbb{U} \\ (2) & \{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \text{ 線性獨立} \end{cases}$ (座標系)

  (基底)

$\dim \mathbb{U} = k$ $\Longrightarrow \mathbb{U} \ni \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$ uniquely (a_i : 座標)

$(\because = b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k \Rightarrow a_i = b_i)$

Example

(1) $\{\mathbb{e}_1, \cdots, \mathbb{e}_n\}$ basis for $\mathbb{C}^n$ standard basis (orthonormal)

(2) $\{|0\rangle, \cdots, |n-1\rangle\}$ 〃 $\mathcal{H}_n$ 〃 〃

(3) $\mathcal{H}_2$: (a) $\{|0\rangle, |1\rangle\} \cong \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

(b) $= \{|+\rangle, |-\rangle\} \cong \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ Hadamard (〃)

$= \left\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}$ Orthogonal (〃)